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On the Weyl Character Formula for $SU(n)$

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Abstract

In this Note, we draw a straight line between the representation theory of $SU(3)$ and the $SU(3)$-classification schemes in particle physics. Our approach is based on that of Weyl, but we have in mind the versions which appear, "in modern dress," in Adams and Bott. Our formulation brings an important part of particle physics into line with two contemporary accounts of compact Lie groups.

1. *The Group SU(3)*

We begin with a celebrated formula due to Weyl (1950, p. 381). Let $SU(3)$ be the group of all 3 x 3 unitary matrices with determinant 1. Let $M(\epsilon)$ = diag $(\epsilon_1, \epsilon_2, \epsilon_3)$ be a diagonal matrix in $SU(3)$. Thus $|\epsilon_1| = |\epsilon_2| = |\epsilon_3| = 1 = \epsilon_1\epsilon_2\epsilon_3$. Let U be an irreducible representation of $SU(3)$ on a finite-dimensional complex vector space, and let χ be its character. Thus $\chi(M) = \mathrm{Tr}(U(M))$, $M \in SU(3)$. Let

$$|\epsilon^r, \epsilon^s, 1| = \begin{vmatrix} \epsilon_1^r & \epsilon_1^s & 1 \\ \epsilon_2^r & \epsilon_2^s & 1 \\ \epsilon_3^r & \epsilon_3^s & 1 \end{vmatrix}$$

where r, s are positive integers. The Weyl formula for the irreducible characters $\chi_{r,s}$ of $SU(3)$ is

$$\boxed{\chi_{r,s}(M(\epsilon)) = |\epsilon^r, \epsilon^s, 1|/|\epsilon^2, \epsilon, 1| \quad r > s}$$

It follows that the irreducible characters are symmetric polynomials in $\epsilon_1, \epsilon_2, \epsilon_3$. Here is a table of five useful characters:

(1) $\chi_{2,1}(M(\epsilon)) = 1$

(2) $\chi_{3,1}(M(\epsilon)) = \epsilon_1 + \epsilon_2 + \epsilon_3$

(3) $\chi_{3,2}(M(\epsilon)) = \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2$

(4) $\chi_{4,2}(M(\epsilon)) = 2 + \epsilon_3\epsilon_2{}^2 + \epsilon_3\epsilon_1{}^2 + \epsilon_1\epsilon_3{}^2 + \epsilon_1\epsilon_2{}^2 + \epsilon_2\epsilon_1{}^2 + \epsilon_2\epsilon_3{}^2$

(5) $\chi_{5,1}(M(\epsilon)) = 1 + \epsilon_1{}^3 + \epsilon_2{}^3 + \epsilon_3{}^3 + \epsilon_3\epsilon_2{}^2 + \epsilon_3\epsilon_1{}^2 + \epsilon_1\epsilon_3{}^2 + \epsilon_1\epsilon_2{}^2 + \epsilon_2 e_1{}^2 + \epsilon_2\epsilon_3{}^2$

Now every symmetric polynomial (with integer coefficients) in $\epsilon_1, \epsilon_2, \epsilon_3$ is a polynomial in the elementary symmetric functions $\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2, \epsilon_1\epsilon_2\epsilon_3 = 1$. This is the mathematical basis of the claim that "the quarks and antiquarks generate all $SU(3)$ multiplets."

2. *The Subgroup* T

The subgroup T of diagonal matrices is clearly commutative, hence its irreducible representations are one-dimensional. We recall that a one-dimensional representation is identical with its character. Let $\chi_{r,s}$ determine the irreducible representation $U_{r,s}$. Now the restriction of $U_{r,s}$ to T is a sum of one-dimensional representations, and the Weyl formula specifies these.

Let us write $\epsilon_k = \exp(2\pi i x_k)$, with x_k real, $k = 1, 2, 3$, and $x_1 + x_2 + x_3 = 0$. Then the representation sending $M(\epsilon)$ to $\epsilon_3\epsilon_2{}^2$ has, as derivative at $1 \in T$, the linear map sending $\mathrm{diag}\,(ix_1, ix_2, ix_3)$ to $x_2 - x_1$. Such a linear map is called a *weight*, and is denoted by its value $x_2 - x_1$. Corresponding to the table of characters in sec. 1, we may construct a table of weights:

(1) 0

(2) x_1, x_2, x_3

(3) $-x_1, -x_2, -x_3$

(4) $0, 0, x_2 - x_1, x_1 - x_2, x_3 - x_2, x_2 - x_3, x_1 - x_3, x_3 - x_1$

(5) $0, 3x_1, 3x_2, 3x_3, x_2 - x_1, x_1 - x_2, x_3 - x_2, x_2 - x_3, x_1 - x_3, x_3 - x_1$

The six nonzero weights in (4) are called *roots*, and are associated with the adjoint representation, whose character is indeed $\chi_{4,2}$.

3. *The* SU(*3*) *Multiplets*

Let Q = charge matrix = $\mathrm{diag}\,(2i/3, -i/3, -i/3)$. Corresponding to the table of weights in sec. 2, we may, by setting $x_1 = 2/3, x_2 = -1/3, x_3 = -1/3$, construct a table of charges:

(1) 0

(2) 2/3, −1/3, −1/3

(3) −2/3, 1/3, 1/3

(4) 0, 0, −1, 1, 0, 0, 1, −1

(5) 0, 2, −1, −1, −1, 1, 0, 0, 1, −1

Note the fractional charges in (2) and (3).

Similarly, we may construct tables for Y and I_3, where Y = hypercharge matrix = diag $(i/3, i/3, -2i/3)$ and I_3 = third-component-of-isospin matrix = diag $(i/2, -i/2, 0)$. We recognize that (2), (3), (4), (5) correspond respectively to quark, antiquark, octet, decuplet. Indeed, we now have enough information to plot the octets and decuplets.

To work with characters, recall that

(i) The character of the direct sum $U_{r,s} \oplus U_{r',s'}$ is the sum $\chi_{r,s} + \chi_{r',s'}$.

(ii) The character of the tensor product $U_{r,s} \otimes U_{r',s'}$ is the product $\chi_{r,s} \cdot \chi_{r',s'}$.

(iii) The character of the dual $U^*_{r,s}$ of $U_{r,s}$ is the complex conjugate $\chi^*_{r,s}$.

(iv) The character of the restriction of $U_{r,s}$ to $SU(2) \subset SU(3)$ is obtained by setting $\epsilon_3 = 1$. This is the branching law (cf. Weyl, 1950, p. 391).

(v) The dimension of $U_{r,s}$ is $\chi_{r,s}(1)$.

4. *The Group* SU(n)

The Weyl formula for $SU(n)$ is

$$\boxed{\chi_{r_1, r_2, \ldots, r_{n-1}}(M(\epsilon)) = |\epsilon^{r_1}, \epsilon^{r_2}, \ldots, \epsilon^{r_{n-1}}, 1| / |\epsilon^{n-1}, \epsilon^{n-2}, \ldots, \epsilon, 1|}$$

where $M(\epsilon) = \operatorname{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in SU(n)$, $r_1, r_2, \ldots, r_{n-1}$ are integers such that $r_1 > r_2 > \cdots > r_{n-1} > 0$, and

$$|\epsilon^{r_1}, \epsilon^{r_2}, \ldots, \epsilon^{r_{n-1}}, 1| = \begin{vmatrix} \epsilon_1^{r_1} \cdots & \epsilon_1^{r_{n-1}} & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \epsilon_n^{r_1} & \epsilon_n^{r_{n-1}} & 1 \end{vmatrix}$$

5. *The Bialternant Symmetric Functions of Jacobi*

We return to $SU(3)$. Let σ_j denote the elementary symmetric function of degree j in $\epsilon_1, \epsilon_2, \epsilon_3$. Thus

$$\sigma_1 = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad \sigma_2 = \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2, \quad \sigma_3 = \epsilon_1\epsilon_2\epsilon_3 = 1$$

The sum of the products of the $\epsilon_1, \epsilon_2, \epsilon_3$, taken j at a time and with unrestricted repetition of any ϵ_i in a product, is called the complete homogeneous symmetric function of degree j and is denoted τ_j. Thus

$\tau_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$

$\tau_2 = \epsilon_1{}^2 + \epsilon_2{}^2 + \epsilon_3{}^2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2$

$\tau_3 = \epsilon_1{}^3 + \epsilon_2{}^3 + \epsilon_3{}^3 + \epsilon_3\epsilon_2{}^2 + \epsilon_3\epsilon_1{}^2 + \epsilon_1\epsilon_3{}^2 + \epsilon_1\epsilon_2{}^2 + \epsilon_2\epsilon_1{}^2 + \epsilon_2\epsilon_3{}^2 + \epsilon_1\epsilon_2\epsilon_3$

The expression

$$\chi_{r,s} = |\epsilon^r, \epsilon^s, 1| / |\epsilon^2, \epsilon, 1|$$

is an example of a *bialternant*. The bialternants were studied by Jacobi, who proved that

$$\chi_{r,s} = \begin{vmatrix} \tau_{s-1} & \tau_{r-1} \\ \tau_{s-2} & \tau_{r-2} \end{vmatrix}$$

By convention, $\tau_0 = 1$, $\tau_j = 0$ if $j < 0$. The dual form of a bialternant is available, and we illustrate with the character $\chi_{4,2}$. The complement of $\{2, 4\}$ in $\{1, 2, 3, 4\}$ is $\{1, 3\}$, and $\{4\text{-}3, 4\text{-}1\} = \{1, 3\}$. Hence

$$\chi_{4,2} = \begin{vmatrix} \sigma_1 \sigma_3 \\ \sigma_0 \sigma_2 \end{vmatrix} = \sigma_1\sigma_2 - 1$$

since, by convention, $\sigma_0 = 1$.

For more on the dual form, consult Aitken (1967). The dual form is of considerable theoretical interest, since it gives $\chi_{r,s}$ *explicitly* as a polynomial in σ_1, σ_2

Our table of characters may now be written as follows:

(1) $\chi_{2,1} = 1$

(2) $\chi_{3,1} = \sigma_1$

(3) $\chi_{3,2} = \sigma_2$

(4) $\chi_{4,2} = \sigma_1\sigma_2 - 1$

(5) $\chi_{5,1} = \tau_3$

In the case of $SU(2)$, the Weyl formula is

$$\chi_r = |\epsilon^r, 1| / |\epsilon, 1| = \tau_{r-1}$$

the complete homogeneous symmetric function in ϵ_1, ϵ_2 of degree $r - 1$. Since $\epsilon_1\epsilon_2 = 1$, one has the more familiar form

$$\chi_r = \epsilon_1^{-(r-1)} + \epsilon_1^{-(r-3)} + \cdots + \epsilon_1^{r-3} + \epsilon_1^{r-1}$$

Since $\chi_r(1) = r$, χ_{2j+1} determines the "spin j" representation of $SU(2)$.

It is worth noting that the denominator in the Weyl formula for $SU(n)$ is the well-known Vandermonde determinant:

$$|\epsilon^{n-1}, \ldots, \epsilon, 1| = \prod_{i<j} (\epsilon_i - \epsilon_j)$$

In this model, an elementary particle is represented by an irreducible T space. A given representation π of $SU(3)$ determines, by restriction to T, an n-tuplet of irreducible T spaces, where $n = \dim\pi$. Note that the "observables" Q, Y, I_3 lie in $\mathbf{t}$, the Lie algebra of T. Of course, $SU(3)$ has rank 2 so that Q, Y, I_3 are linearly dependent. Now, giving the representation space of π an invariant inner product, we have

$$\pi\colon SU(3) \to U(n)$$

$$\pi|T\colon\ T \longrightarrow U(1) \times U(1) \times \cdots \times U(1) \subset U(n)$$

$$\pi'|\mathbf{t}\colon\ \mathbf{t} \longrightarrow \mathbb{R}\oplus \cdots \oplus\mathbb{R} = \mathbb{R}^n$$

Thus π assigns to each observable in $\mathbf{t}$ an n-tuplet of real quantum numbers. For example, $\pi'(Q)$ assigns, to each particle in the octet, its charge.

It is of interest to locate Q, Y, I_3 on the Stiefel diagram of $SU(3)$ (Adams, 1969, p. 104).

The baryon octet and decuplet behave beautifully with respect to the Weyl group. The Weyl group of $SU(3)$ is the symmetric group S_3.

(i) Octet. The Weyl group permutes amongst themselves the six particles located at unit distance from O; S_3 leaves alone the two remaining particles.

(ii) Decuplet. The Weyl group permutes amongst themselves the three particles located at distance 2 from 0; permutes amongst themselves the six particles located at distance 1 from 0, leaves alone the particle at the origin.

To illustrate the branching law in Sec. 3, let us recover the "isospin multiplets."

(i) Octet. Set $\epsilon_3 = 1$. We thereby restrict the $SU(3)$ action to $SU(2)$. The resulting symmetric polynomial is

$$1 + 2(\epsilon_1 + \epsilon_2) + (\epsilon_1^2 + 1 + \epsilon_2^2)$$

corresponding to

$$1 \oplus 2D_{1/2} \oplus D_1$$

thus yielding an isospin singlet, two doublets, and a triplet.

(ii) Decuplet. Set $\epsilon_3 = 1$. We get

$$(\epsilon_1^3 + \epsilon_1 + \epsilon_2 + \epsilon_2^3) + (\epsilon_1^2 + 1 + \epsilon_2^2) + (\epsilon_1 + \epsilon_2) + 1$$

corresponding to

$$D_{3/2} \oplus D_1 \oplus D_{1/2} \oplus 1$$

thus yielding an isospin singlet, doublet, triplet, quadruplet.

The diagrams for the baryon octet and decuplet are familiar and will not be reproduced here. We emphasize that the particles themselves are represented by irreducible T-spaces. For the octet, the neutron and proton are represented by the irreducible T-spaces $\epsilon_1\epsilon_2^2$ and $\epsilon_2\epsilon_1^2$, respectively. For the decuplet, the Ω^- particle is represented by ϵ_3^3.

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